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DEVIATION OF THE WAVES IN AN INHOMOGENEOUS MEDIUM

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Abstract. Using the formula found by Noorbala and Sepehrinia, the wave deviation in an inhomogeneous medium with continuous variation of propagation velocity is deduced. For electromagnetic waves (light) that propagate in the gravitational field, the deduced deviation is identical to that calculated from General Relativity. The method and their consequences are a good pedagogical example that verifies the Noorbala-Sepehrinia’s formula as well as the mechano-optics analogy (Hamilton’s principle/principle of stationary action and Fermat’s principle) for the bodies movement in the gravitational field.

Keywords: Noorbala-Sepehrinia’s formula, Hamilton’s principle, gravitational deviation of light.

1. Introduction

In a paper (Noorbala and Sepehrinia, 2016), Noorbala and Sepehrinia (N-S) found a formula which relate the refractive index n to the angle θ of incidence (the angle between incident light ray and normal at the constant index

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surface) for the case when the speed of the light varies continuously within a medium. This new relation is different to that one of Snell' law.

We will use it in our paper to compute the deviation of the wave (particularly, the light) when it travels an inhomogeneous medium (Sarbort and Tyc, 2012; Born and Wolf, 1970; Evans and Rosenquist, 1986). For this kind of medium we will assume that the refractive index depends only on the radius $n(r)$. Our result is like that one obtained in General Relativity for an isotropic metrics (Lerner, 1997; de Felice, 1971; Simaciu and Ionescu-Pallas, 1996).

All our derivations which we will perform in the following rows may be an opportunity to make a pedagogical practice to use the analogy between mechanics and optics, in order to prove the N-S relation and to study the motion of a body in a gravitational field (Evans and Rosenquist, 1986; Fasano and Marmi, 2006; Evans *et al.*, 1996).

2. N-S Relation for a Medium with Refractive Index $n(r)$

Let's emphasize an inhomogeneous medium with refractive index depending on the radius as

$$n(r) = \exp\left(\frac{N}{r^p}\right), N > 0,\ p = \frac{1}{2}, 1, \frac{3}{2}, 2, ..., . \tag{1}$$

The above dependency is suggested by Rastall-Yilmaz-Rosen metrics (Rastall, 1968a; Rastall, 1968b; Yilmaz, 1958; Rosen, 1974). We will use it since it allows us to obtain an approach for the bending of the waves when the refractive index is of the form

$$n(r) = \exp\left(\frac{N}{r^p}\right) \simeq 1 + \frac{N}{r^p},\ N > 0,\ \frac{N}{r^p} << 1. \tag{2}$$

For $p = 1$ and $N = r_g = 2Gm/c^2$ (where m is the mass of a body placed in the origin of the reference frame), Eq. (2) becomes

$$n(r) = 1 + \frac{r_g}{r}. \tag{3}$$

This gravitational index of refraction is compatible with the Schwarzschild metric (Møller, 1955, §Ch. 123).

According to the equation (10) of the paper of Noorbala and Sepehrinia (Noorbala and Sepehrinia, 2016), for continuous and inhomogeneous medium the law of refraction is

$$\sin\theta \frac{d(n\sin\theta)}{ds} = -n\cos\theta\left(\frac{d\hat{n}}{ds}\hat{\upsilon}\right). \tag{4}$$

The surface of a medium where the refractive index is constant is a spherically one.

As shown in Fig. 1, $\vec{r}(r,\varphi)$ is the position vector in polar coordinates, $\hat{n}$ is the normal to the surface, $\hat{\upsilon}$ is the direction of the vector $\vec{\upsilon} = d\vec{r}/ds$, θ is the angle between the vectors $\hat{n}$ and $\hat{\upsilon}$ and δ is the bending angle.

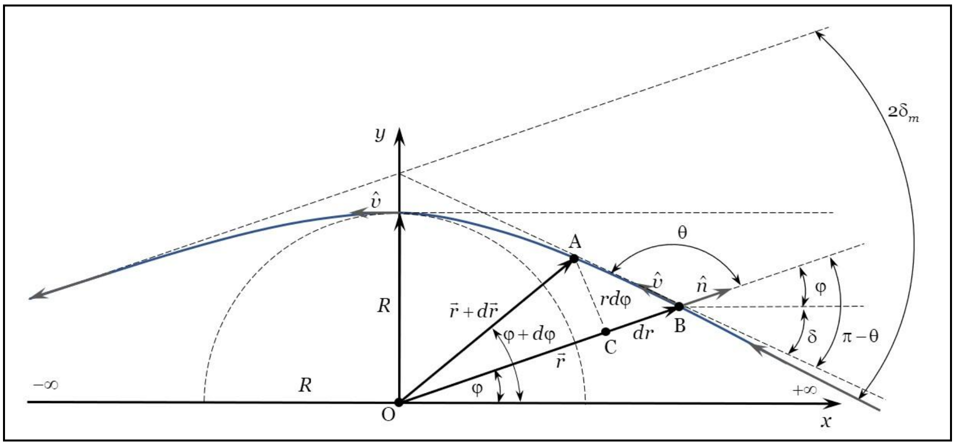

Fig. 1 – Position vector in polar coordinates.

Since the refractive index depends only on the length of the position vector, according to Eq. (2), the normal to the surface $\hat{n}$, the position vector and the gradient of the index $\nabla n = (\partial n/\partial r)(\vec{r}/r) = (\partial n/\partial r)\hat{n}$ are in a parallel direction. It follows

$$\left|\frac{d\hat{n}}{ds}\right| = \left|\frac{d}{ds}\left(\frac{\vec{r}}{r}\right)\right| = \frac{d\varphi}{ds}, \tag{5}$$

where φ is, according to Fig. 1, the angular variable in polar coordinates (r,φ) in the plane of the wave path

The vector $d\hat{n}/ds$ is perpendicular to $\hat{n} = \vec{r}/r$ and

$$\frac{d\hat{n}}{ds}\hat{\mathrm{v}} = \frac{d\varphi}{ds}\cos\left(\frac{\pi}{2} - \theta\right) = \frac{d\varphi}{ds}\sin\theta\,. \tag{6}$$

According to Fig. 1, to Fig. 2 and to the notations adopted in (Sarbort, 2012) and (Noorbala and Sepehrinia, 2016) one can establish the following correspondences

$$dl \leftrightarrow ds = \sqrt{dr^2 + r^2 d\varphi^2}, \quad \alpha \leftrightarrow \theta\,, \tag{7}$$

When substitute Eq. (6) in Eq. (4), it follows

$$\sin\theta \frac{d(n\sin\theta)}{ds} = -n\frac{d\varphi}{ds}\cos\theta\sin\theta\,, \tag{8a}$$

or

$$d\left(n\sin\theta\right)=-n\cos\theta d\varphi, \tag{8b}$$

which is the N-S relation mentioned in the rows above.

3. N-S Relation and the Law of Conservation of Angular Momentum

In what it follows we will derive the law of conservation of angular momentum and we will prove the compatibility of it with N-S relation for the case when the refractive index depends on the length of the position vector.

The path of the light has the constant (Sarbort and Tyc, 2012; Born and Wolf, 1970; Evans and Rosenquist, 1986). According to the Eq. (1) of the paper (Sarbort and Tyc, 2012), this constant is

$$L=rn\left(r\right)\sin\theta. \tag{9}$$

This constant corresponds to the conservation of the length of the angular momentum of a photon $\vec{L}=\vec{r}\times\vec{p}$ (see Appendix 1).

At $\theta=\pi/2$ the radius becomes $r\left(\pi/2\right)=R$. Using this radius and Eqs. (1) and (9), the constant becomes

$$L=Rn\left(R\right)=R\exp\left(\frac{N}{R^{p}}\right). \tag{10}$$

One can differentiate (9), and so

$$d\left(n\sin\theta\right)=-\frac{dr}{r}n\sin\theta. \tag{11}$$

We will use the Fig. 1 of this paper and the Fig. 2 from (Sarbort, 2012) to establish the following relations

$$\sin\theta=\frac{rd\varphi}{ds},\quad \cos\theta=\frac{dr}{ds}. \tag{12}$$

When substituting relations (12) within the right side of (11) one can obtain the N-S relation (8b). It is not surprisingly that these two relations are compatible.

The N-S relation was derived tacking account of the generalized Fermat's principle (Noorbala and Sepehrinia, 2016) which is in fact the Hamilton's principle for light (Born and Wolf, 1970; Evans and Rosenquist, 1986). Using the Hamilton's principle one derives the angular momentum conservation (Fasano and Marmi, 2006).

4. The Bending of the Light Traveling in an Inhomogeneous Medium

We will apply N-S relation for computing the deviation of ray of light which is parallel to Ox and start from a point of coordinates $r\to+\infty, \varphi\to 0$ an it is directed toward the point of coordinates $r = R, \varphi = \pi/2$ (see Fig. 1). Here R is the minimum length of the position vector related to the reference frame. Here the bending angle is zero, $\delta = 0$, and $\theta = \pi/2$, since these two angles are related by

$$\varphi + \delta + \theta = \pi \tag{13a}$$

and

$$d\theta = -d\varphi - d\delta \tag{13b}$$

When express Eq. (7b) using *r, n* and φ (Appendix 2, Eq. (A2.3)), then

$$d\delta = \frac{r}{n}\frac{dn}{dr}d\varphi. \tag{14}$$

Differentiating Eq. (1), yields

$$\frac{dn}{dr} = -\frac{pN}{r^{p+1}}n\,. \tag{15}$$

Replacing Eq. (15) in Eq. (14), one obtains

$$d\delta = -\frac{pN}{r^p}d\varphi\,. \tag{16}$$

To integrate the Eq. (16) it is necessary to find out how φ depends on r. According to (Sarbort and Tyc, 2012; Born and Wolf, 1970; Evans and Rosenquist, 1986), the path of the light in a medium is depicted by (Eq. (2) from (Sarbort and Tyc, 2012))

$$d\varphi = \pm\frac{Ldr}{r\sqrt{n^2r^2 - L^2}}, \tag{17}$$

where L is the constant from Eqs. (9) and (10).

Then we will do two replacements: first Eqs. (1) and (10) in Eq. (17)

$$d\varphi = \pm\frac{R\exp\left(N/R^p\right)dr}{r\sqrt{r^2\exp\left(2N/r^p\right) - R^2\exp\left(2N/R^p\right)}} \tag{18}$$

and second Eq. (18) in Eq. (16)

$$d\delta = \mp\frac{pNR\exp\left(N/R^p\right)dr}{r^{p+1}\sqrt{r^2\exp\left(2N/r^p\right) - R^2\exp\left(2N/R^p\right)}}\,. \tag{19}$$

Integrating Eq. (19) between $r \to +\infty$ and $r = R$ we obtain the maximum deviation $\delta_m = \delta\left(r \to +\infty\right)$

$$\int_{\delta_m}^{0} d\delta = \mp \int_{+\infty}^{R} \frac{pNR\exp\left(N/R^p\right)dr}{r^{p+1}\sqrt{r^2\exp\left(2N/r^p\right) - R^2\exp\left(2N/R^p\right)}} . \tag{20}$$

Here, the integral (20)

$$\delta_m = \mp \frac{\sqrt{\pi}N\Gamma\left(\frac{p+1}{2}\right)\mathrm{e}^{\frac{N}{2R^p}}}{R^p\Gamma\left(\frac{p}{2}\right)} \cong \mp \frac{\sqrt{\pi}N\Gamma\left(\frac{p+1}{2}\right)\left(1+\frac{N}{2R^p}+\frac{N^2}{8R^{2p}}\right)}{R^p\Gamma\left(\frac{p}{2}\right)}, \tag{21}$$

is an analytical formula using WolframAlpha or Wolfram Programing Lab (WolframAlpha).

5. The Deviation of the Light in a Gravitational Field

Assuming a gravitational field of a body with mass *m*, the refractive index can be expressed like (3), for the first-order isotropic Schwarzshild metric (Lerner, 1997; de Felice, 1971; Simaciu and Ionescu-Pallas, 1996). Therefore, the maximum deviation, with $p = 1$ and $N = r_g$ in Eq. (21), is

$$\delta_m \cong \pm \frac{r_g}{R} . \tag{22}$$

The entire deviation, which occurs when the wave travels from $r \to +\infty$ to $r \to -\infty$, and passing at least distance *R*, is

$$\delta_g = 2\delta_m = \frac{2r_g}{R} . \tag{23}$$

This result is just the relativistic deviation (Lerner, 1997; Møller, 1955). That is, an inhomogeneous optic medium with a refractive index of the form (3) behaves like a spherical lens and this lens mimics a gravitational field.

One can rich the same result using lensmaker's Eq. (36) from the paper (Born and Wolf, 1970, §Ch. 4.4) for a spherical lens with radii of curvature $R_1 = R_2 = R$ and refractive index $n_p(R) = 1 + N/R^p$

$$\frac{1}{f_p} = \left[n(R) - 1\right]\frac{2}{R} = \frac{2N}{R^{p+1}} . \tag{24}$$

Then, the entire deviation for a parallel ray to the direction and which pass through a point of coordinates $r = R$ is

$$\delta_p \simeq \tan\delta_p = \frac{R}{f_p} = \frac{2N}{R^p}. \tag{25}$$

For $p = 1$ and $N = r_g$ the entire deviation is

$$\delta_1 = \delta_g = \frac{R}{f_g} = \frac{2r_g}{R}. \tag{26}$$

and this is just the relativistic deviation from Eq. (23).

6. Conclusions

According to the general relativity, half of the bending of the light in a gravitational field depends on the curvature of the space and the other half to the variation of the velocity of the light along its path (Møller, 1955).

An optical approach has to assume that the deviation of the waves, no matter of their kind (Simaciu *et al.*, 2018), and for an inhomogeneous medium, $n(r)$, depends on the optical properties of the medium, as we shown in this paper.

Since in the classic physics the bending of the path it is assumed to the action of a force, then a gravitational force (*i.e.* which is directly proportional to the mass and inversely proportional to the distance) is also the effect of a refractive index which depends on the position vector. Therefore, if will be able to assume a scenario of becoming inhomogeneous the medium around a particle, then we will be able to get a phenomenological-causal approach of the gravitational interaction into the electromagnetic world, *i.e.* the world where the main interaction is the electromagnetic one.

For a refractive index from Eq. (3), the acceleration is directly proportional to r_g / r^2, *i.e.* a gravitational type of acceleration.

This kind of approach had succeed to apply the mechano-optics of the gravitational interaction to the research of the light and also to the particles with rest mass (Evans *et al.*, 1996).

Appendix 1

According to the mechano-optics the angular momentum of a photon which moves in a gravitational field is

$$\left|\vec{L}_m\right| = \left|\vec{r} \times \vec{p}\right| = rp\sin\theta. \tag{A1.1}$$

Its momentum is

$$p = m\upsilon = \hbar k. \tag{A1.2}$$

where k is the wave number,

Using the angular speed ω, we may express the mass of the photon as $m = \hbar\omega/c$, the speed as $\upsilon = c/n$ and the momentum as

$$p = \frac{\hbar\omega}{c}\upsilon = \hbar k = \hbar\frac{n\omega}{c}. \quad \text{(A1.3)}$$

Replacing the Eqs. (A1.3) and (A1.2) in Eq. (A1.1), yields

$$\left|\vec{L}_m\right| = \left(\frac{\hbar\omega}{c}\right) rn\sin\theta = \left(\frac{\hbar\omega}{c}\right) L. \quad \text{(A1.4)}$$

Since the angular momentum is a constant during its motion and also $\hbar\omega/c$ (the angular speed doesn't change during the travelling through the medium), then the following parameter

$$L = \frac{c\left|\vec{L}_m\right|}{\hbar\omega} = rn(r)\sin\theta \quad \text{(A1.5)}$$

is a constant.

Appendix 2

Eq. (8b) can be put also in the form

$$dn\sin\theta + n\cos\theta d\theta = -nd\varphi\cos\theta. \quad \text{(A2.1)}$$

Replacing $d\theta$ from Eq. (13b) in Eq. (A1.1), one obtains

$$d\delta = \frac{dn}{n}\frac{\sin\theta}{\cos\theta}. \quad \text{(A2.2)}$$

Finally using Eq. (12) in Eq. (A1.2), it follows

$$d\delta = \frac{r}{n}\frac{dn}{dr}d\varphi. \quad \text{(A2.3)}$$

NOTE. This paper has been uploaded on the arXiv platform: 1810.07029 (physics.gen-ph).

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DEVIAȚIA UNDELOR ÎNTR-UN MEDIU NEOMOGEN

(Rezumat)

Folosind formula obținută de Noorbala și Sepehrinia, deducem deviația undelor într-un mediu neomogen cu variație continuă a vitezei de propagare. Pentru undele electromagnetice (lumina) care se propagă în câmpul gravitațional, abaterea dedusă este identică cu cea calculată în Relativitatea Generală. Metoda și consecințele acesteia sunt un bun exemplu care verifică formula Noorbala-Sepehrinia precum și analogia mecano-optică (principiul lui Hamilton/principiul acțiunii staționare și principiul lui Fermat) pentru mișcarea corpurilor în câmpul gravitațional.